Memory effects in the statistics of interoccurrence times between large returns in financial records

Mikhail I. Bogachev and Armin Bunde
Institut für Theoretische Physik III, Justus-Liebig-Universität Giessen, 35392 Giessen, Germany
(Received 11 February 2008; revised manuscript received 18 June 2008; published 22 September 2008)

We study the statistics of the interoccurrence times between events above some threshold Q in two kinds of multifractal data sets (multiplicative random cascades and multifractal random walks) with vanishing linear correlations. We show that in both data sets the relevant quantities (probability density functions and the autocorrelation function of the interoccurrence times, as well as the conditional return period) are governed by power laws with exponents that depend explicitly on the considered threshold. By studying a large number of representative financial records (market indices, stock prices, exchange rates, and commodities), we show explicitly that the interoccurrence times between large daily returns follow the same behavior, in a nearly quantitative manner. We conclude that this kind of behavior is a general consequence of the nonlinear memory inherent in the multifractal data sets.

DOI: 10.1103/PhysRevE.78.036114 PACS number(s): 89.65.Gh, 05.40.−a, 89.75.Da, 05.45.Tp

I. INTRODUCTION

In recent years, the occurrence of rare (extreme) events has attracted much attention [1–3]. Usually, rare events with magnitudes considerably exceeding the average magnitude, have been considered as independent, since the typical time span between them is very large. In recent years, however, there is growing evidence that this assumption is not always true. In particular, in financial markets large volatilities seem to cluster [4], and in paleoclimate records a clustering of large river flows or high temperatures has also been observed [5]. To quantify the occurrence of rare events one usually considers the time interval between successive events above (or below) some threshold Q. One is interested in the probability distribution function of these return intervals as well as in their long-term dependencies (autocorrelation function, conditional return periods, etc.). In numerical treatments, one usually considers not too large thresholds Q where the statistics of the return intervals is good, and then tries to extrapolate the results towards very large thresholds where the statistics, by definition, is poor.

For independent data sets, the return intervals are independent and (according to Poisson statistics) exponentially distributed. Clustering of rare events indicates a certain memory in the return intervals, and indeed recent studies have shown that this kind of memory is a consequence of long-term dependencies in the time series itself [5–8], which occur, for example in climate [5,9] and physiological records [10–12], as well as in time series demonstrating human behavior, including economic records [13–15], teletraffic in large networks [16], and crowd behavior [17].

Long-term memory can be either (i) linear, (ii) nonlinear, or (iii) both linear and nonlinear. In the first case, which is often referred to as "monofractal" the (linear) autocorrelation function $C_x(s)$ of the data decays with time s by a power law, $C_x(s) \sim s^{-\gamma}$, $0 < \gamma < 1$, and the exponent γ fully describes the correlations within the record. In this case, the return intervals are long-term correlated in the same way as the original record, and its distribution is characterized at large scales by a stretched exponential with exponent γ, and at short scales by a power law with exponent $\gamma - 1$ [5–7]. It has been shown that those features can be observed in long climate records [5] as well as in the volatility of financial records [4], even though the volatility also contains nonlinear memory and thus belongs to the case (iii).

In the second case, where the record is "multifractal," the linear autocorrelation function $C_x(s)$ vanishes for $s > 0$ and nonlinear (multifractal) correlations, which cannot be described by a single exponent, characterize the record. Examples are precipitation records or the returns in financial records. Using a variant of the multiplicative random cascade (MRC) model, we have shown recently that the nonlinear correlations inherent in such time series provide a pronounced effect on the statistics of return intervals, leading to probability density functions (PDFs), autocorrelation functions, and conditional return periods that exhibit power-law behavior, in marked contrast to the independent and monofractal long-term-correlated data series [8]. The exponents of these power laws significantly depend on the selected threshold, i.e., different behavior is observed for return intervals between smaller and larger events. Therefore, no straightforward extrapolation of the laws governing the return intervals between smaller events can be done for quantifying intervals between larger events. We have also provided some evidence that these features characterize the International Business Machines (IBM) stock-price record [8].

The aim of this paper is twofold. First, we show that qualitatively similar results can be obtained for data series generated by a different kind of model known as the multifractal random walk (MRW), proposed in [18]. This demonstrates that the observed behavior is a universal consequence of the nonlinear correlations inherent in the multifractal data, and not an artifact of a certain determinism preserved in the MRC creation procedure. Second, we extend our studies to a large number of various financial records, in this way providing the empirical evidence of the universality of the observed behavior on financial markets. We also compare the empirical results with both MRC and MRW models to elucidate whether the first or the second model fits better to the empirical evidence and therefore appears more suitable for making predictions of the dynamics of certain quantities.

The paper is organized as follows. In Sec. II we briefly review long-term dependencies and multifractalilty in financial time series. In Sec. III we describe the two procedures (MRC and MRW) to generate multifractal data with vanishing linear correlations. In Sec. IV we discuss the return interval statistics, including the PDF, autocorrelation function (ACF), and conditional return periods for both models, and discuss the similarities and the differences between them. In Sec. V we study the same quantities in financial markets, including market indices and currency exchange rates, as well as stock and oil prices. Finally, in Sec. VI, we summarize our results.

II. LONG-TERM MEMORY IN FINANCIAL MARKETS

In financial markets, the central quantity is the return R_i after the ith unit trading period (which might differ from minutes to years), related to the price P_i by

$$R_i = \frac{P_i - P_{i-1}}{P_{i-1}}. \quad (1)$$

Here we concentrate on daily closing prices P_i, where i denotes subsequent bank days. By definition, positive returns characterize gains and negative returns characterize losses. One of the reasons to consider returns instead of prices is that the time series of the prices is usually nonstationary (in most cases, the same holds for the increments $P_i - P_{i-1}$ on long time lags, due to inflation).

There is empirical evidence that the linear ACF

$$C_R(s) = \frac{1}{\sigma_R^2(L-s)} \sum_{i=1}^{L-s} (R_i - \langle R\rangle)(R_{i+s} - \langle R\rangle) \quad (2)$$

of price returns on the open markets vanishes for s above some very short time scale ($s=1$ day), that expresses the latency of the market reaction to new information. We have confirmed this kind of behavior for all records considered here. For the ACF of the IBM stock-price record, we refer to [8]. The absence of linear correlations is consistent with the efficient market hypothesis, that all the available information is instantly processed when it reaches the market and immediately reflected in the prices of the assets traded [19,20].

An important quantity that allows one to quantify nonlinear memory in financial markets is the so-called "structure function" $S_q(s)$ [21]. To calculate S_q, we first subdivide the series of returns R_i into N_s nonoverlapping windows of size s. Next, the absolute values of the local sums in every window k are calculated,

$$Y_k = \left| \sum_{i=(k-1)s+1}^{ks} R_i \right|, \quad (3)$$

and $S_q(s)$ is obtained via

$$S_q(s) = \frac{1}{N_s} \sum_{k=1}^{N_s} Y_k^q. \quad (4)$$

Since the mean value of the returns $\Sigma_{i=1}^N R_i = 0$ vanishes, $S_q(s)$ is equal to the central moment, which is a generalization of the variance of Y_k. For $q=2$, it reduces to the normal variance.

For long-term-dependent data, the structure functions scales as $S_q(s) \sim s^{qh(q)}$, where $h(q)$ is the generalized Hurst exponent, which is constant for monofractal data, and q dependent for multifractal data sets [22,23]. $h(q)$ is directly related to the scaling exponent $\tau(q)$ defined by the standard partition-function-based multifractal formalism [22,24], via $\tau(q) = qh(q) - 1$ [25].

Using various methods that allow estimation of $h(q)$ from a given time series also in the presence of trends [25,26], the multifractal character of the price returns has been shown explicitly in a number of studies (see, for example, [27–29]). Also some particular parameters related to the structure function have been introduced, such as the intermittency parameter $S_1 = -[dh(q)/dq]|_{q=1}$ [30], which is the tangent to the $h(q)$ curve at $q=1$.

In addition, much effort have been undertaken to introduce multifractal models for describing the processes in financial markets [31–38]. Most of these studies dealt with time series on the global market, including some integral market indices (e.g., S&P500, CAC40), important stocks (e.g., IBM), and exchange rates of world currencies (e.g., Deutsche mark vs U.S. dollar) that have a sufficiently long history of trading. Also, the multifractal properties of the absolute returns (volatility) were studied earlier in [27,33,39,40].

A different way of quantifying nonlinear memory is via the ACF of the absolute returns to different powers q, $|R_i|^q$, where pronounced correlations on long time lags have been found, first for $q=1$ [41], and later for general q [42].

III. GENERATION OF MULTIFRACTAL DATA SERIES

There are several ways to generate multifractal data. Here we concentrate on two of those that allow us to create multifractal data series with vanishing linear correlations. The first algorithm is a variant of the multiplicative random cascade process, described, e.g., in [22,43–45]. In this process [8], the data set is obtained in an iterative way, where the length of the record doubles in each iteration. We start with the zeroth iteration $n=0$, where the data set (x_i) consists of one value, $x_1^{(n=0)}=1$. In the nth iteration, the data $x_i^{(n)}$, $i=1,2,\ldots,2^n$, are obtained from

$$x_{2l-1}^{(n)} = x_l^{(n-1)} m_{2l-1}^{(n)} \quad \text{and} \quad x_{2l}^{(n)} = x_l^{(n-1)} m_{2l}^{(n)}, \quad (5)$$

where the multipliers m are independent and identically distributed (i.i.d.) random numbers with zero mean and unit variance (see Fig. 1). The data sets are characterized by a vanishing autocorrelation function, i.e., $C_x(s)=0$ for $s>0$ [8].

The second algorithm is the multifractal random walk proposed in [18]. In this algorithm, first we generate a record a_i, $i=1,\ldots,N$, whose power spectrum decays as $1/f$ ("$1/f$ noise"). Next, we exponentiate these numbers and multiply

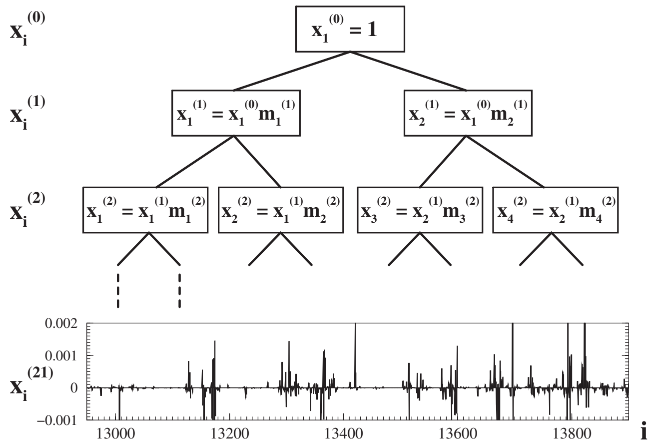

FIG. 1. Illustration of the iterative random cascade process. After each iteration the length of the generated records is doubled and after $n=21$ iterations the multifractal set consists of $L=2^{21}$ numbers. A subset is shown in the bottom panel.

them by Gaussian random numbers b_i, providing the resulting multifractal series x_i (see Fig. 2),

$$x_i = (e^{a_i})b_i. \tag{6}$$

Both models create data series with symmetric distribution characterized by log-normal tails (see Fig. 3). We wish to note that, in general, $h(q)$ is influenced not only by linear and nonlinear correlations, but also by the heavy tails in the distribution of the simulated data, which is less pronounced in financial records. There are several ways to elucidate the effect of the memory inherent in a time series on $h(q)$. The first way is to exchange the data rankwise by non-heavy-tailed distributed data, e.g., Gaussian [8]. The memory is conserved in this exchange process, but the distributional effect on $h(q)$ is eliminated this way. The second way is to shuffle the data. After shuffling, only the distributional effect is preserved in $h(q)$ [25], which can be estimated in this way.

In the more general case (for example, when studying volatilities in financial markets), when a record contains both linear and nonlinear correlations, it is possible to distinguish

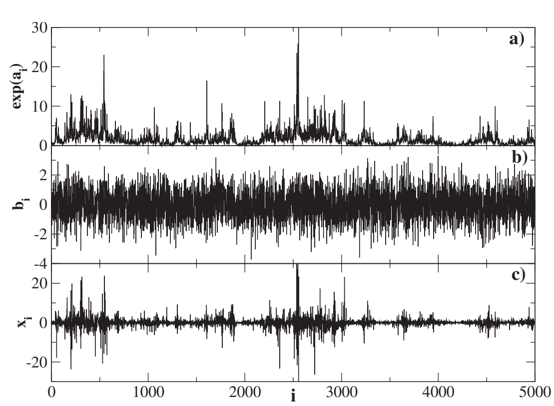

FIG. 2. Illustration of the multifractal random walk generation procedure: the exponentiated $1/f$ noise a_i (a) is multiplied by Gaussian random numbers b_i (b), resulting in the multifractal series x_i (c).

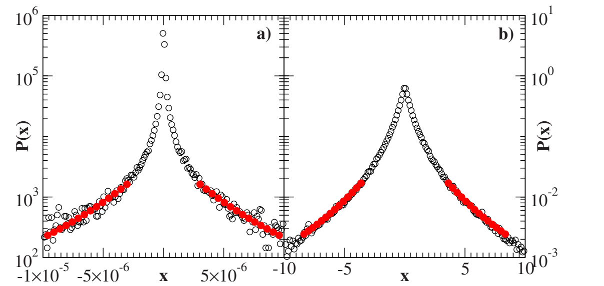

FIG. 3. (Color online) PDFs of the simulated data series, created by (a) MRC and (b) MRW models. The open symbols provide a numerical estimate; the full symbols are the best log-normal fits for the tails of the distribution.

between them by phase randomization. It has been proved that after this procedure only linear correlations remain in the record, since the power spectrum is preserved [46].

IV. RETURN INTERVALS IN THE SIMULATED MULTIFRACTAL DATA SERIES

In the following, we are interested in the statistics of the interoccurrence times, or return intervals r_i, between events above some threshold Q in both the MRC and the MRW models. For an illustration of the procedure of extracting the return interval series from a data series, see Fig. 4.

For a given record, there is a one-by-one correspondence between the threshold Q and the mean return interval (or return period) R_Q, $R_Q = 1/\int_Q^\infty P(x)dx$, where $P(x)$ is the distribution of the data. By fixing R_Q instead of Q, return interval statistics remain unchanged, when the rankwise exchange procedure described above is applied. Accordingly, return interval statistics depend solely on the memory inherent in the data, and hence can be used as an effective instrument for quantifying such memory, independent from a multifractal analysis of the data.

First, we consider the distribution density of the return intervals $P_Q(r)$ for different thresholds Q for the MRC model. As shown in Fig. 5(a), $P_Q(r)$ scales as [8]

$$P_Q(r) \sim (r/R_Q)^{-\delta(Q)}. \tag{7}$$

Deviations from the power-law behavior occur only at very large scales $r > 30R_Q$, and can probably be assigned to finite-size effects. For small R_Q values, there are also some deviations on small scales, which probably are caused by discreteness effects, since the smallest return interval in the considered discrete model is a unit time interval. In the intermediate regime (which covers up to four orders of magnitudes), a power law provides an excellent fit for the PDF

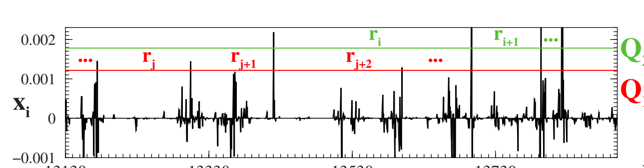

FIG. 4. (Color online) Extraction of the return interval sequences of events above thresholds Q_1 and Q_2 from a data series.

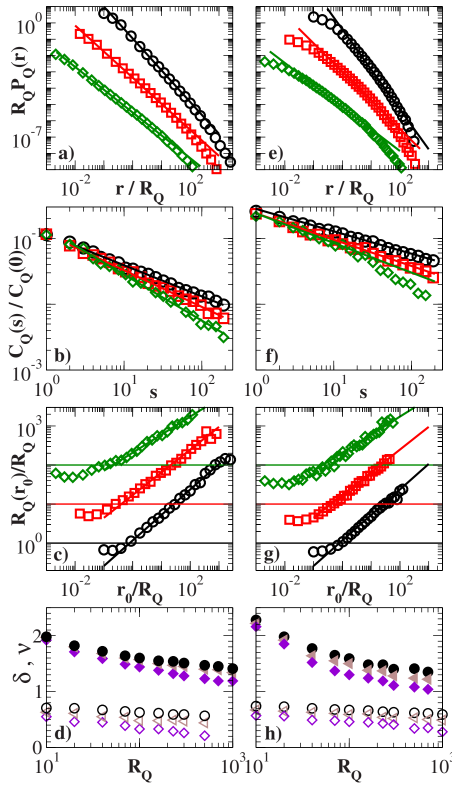

FIG. 5. (Color online) Statistics of the return intervals for the MRC (left column) and MRW (right column) data series. (a) PDFs of the return intervals for the MRC model, for three different quantiles, R_Q=10 (○), 70 (□), and 500 (◇), rescaled with R_Q. Full straight lines represent the best power-law fits; to avoid overlapping, the distributions for R_Q=70 and 500 are shifted downward by factors of 10^2 and 10^4, respectively. (b) ACFs of the return intervals for the same quantiles as in (a). (c) Conditional return periods $R_Q(r_0)$ divided by R_Q versus r_0/R_Q, for the same quantiles as above, with best fits; results for R_Q=70 and 500 are shifted upward by factors of 10 and 100, respectively. (d) Exponents $\delta(Q)$ and $\nu(Q)$ for the PDFs (filled symbols) and the conditional return periods (open symbols) plotted versus R_Q for different system sizes $L=2^{26}$ (○), 2^{21} (◁), and 2^{16} (◇). The panels from (e) to (h) present the same quantities for the MRW data series. All the results in (a)–(c) and (e)–(g) have been obtained for the system size $L=2^{21}$ and averaged over 150 random configurations. Panels from (a) to (d) are adapted from [8].

[see Fig. 5(a)], but with different exponents for different quantiles, in particular δ=−1.95, −1.6, and −1.4 for R_Q=10, 70, and 500, respectively. Accordingly, there is no scaling, and as a consequence the occurrence of extremes cannot be estimated straightforwardly from the occurrence of smaller events.

Qualitatively similar results are obtained for the MRW model [Fig. 5(e)], despite more pronounced deviations from the power law at small and large r/R_Q values. We think that these deviations are a consequence of the multiplicative noise inherent in the MRW model. The dependence of the exponents on R_Q appears to be significantly stronger than in the MRC model, with δ=−2.22, −1.55, and −1.24 for R_Q =10, 70, and 500, respectively.

Next, we consider the correlations among the return intervals. To this end, we consider the ACF of the return interval series, defined as

$$C_Q(s) = \frac{1}{\sigma_r^2(L_Q - s)} \sum_{i=1}^{L_Q - s} (r_i - \langle r \rangle)(r_{i+s} - \langle r \rangle). \qquad (8)$$

Again, for both models under consideration, we find power-law decay of $C_Q(s)$,

$$C_Q(s) \sim s^{-\beta(Q)}, \qquad (9)$$

demonstrating the presence of long-term memory even in the absence of linear correlations in the original data set, in agreement with our recent findings for the MRC model [8]. The exponent β exhibits a slight dependence on the size of the quantile Q, such that the intervals between smaller events (e.g., R_Q=10) appear to be more strongly correlated (i.e., show a smaller exponent β) than the intervals between larger events (e.g., R_Q=500). For the MRC model, the best fits are β=0.46, 0.49, and 0.56 for R_Q=10, 70, and 500, respectively [see Fig. 5(b)].

Figure 5(f) shows that for the MRW model a slower decay of the ACFs appears for all return periods, with β=0.34, 0.4, and 0.45 for the same R_Q values as for the MRC model. In addition, the absolute values of the normalized ACF appeared to be larger for the MRW model. It seems that also for $C_Q(s)$ finite-size effects are more pronounced in the MRW than in the MRC model.

To further quantify the memory among the return intervals, we next consider the conditional return intervals, i.e., we regard only those intervals whose preceding interval is of a fixed size r_0. For both models considered, for r_0 values exceeding the return period R_Q, the conditional return period $R_Q(r_0)$, defined as the mean value of all return intervals following the return intervals with a certain r_0 value, increases by a power law,

$$R_Q(r_0) \sim r_0^{\nu(Q)} \quad \text{for } r_0 > R_Q, \qquad (10)$$

in agreement with our recent findings for the MRC model [8]. It is interesting that the exponents $\nu(Q)$ are identical for both models, with ν=0.63, 0.53, and 0.49 for R_Q=10, 70, and 500, respectively [see Figs. 5(c) and 5(g)].

Finally, we studied the dependence of the exponents $\delta(Q)$ and $\nu(Q)$ on both the global return period R_Q and the system size L. For both models, it seems that, for all system sizes N, both $\delta(Q)$ and $\nu(Q)$ decay logarithmically with R_Q. Again, for the MRW model, there are earlier and stronger deviations from this kind of behavior. The R_Q dependence of the exponent δ is more pronounced, while the system size dependence is roughly of the same level. It seems that for large system sizes both $\delta(Q)$ and $\nu(Q)$ dependencies demonstrate the tendency of collapsing into limiting curves [see Figs. 5(d) and 5(h)].

V. RETURN INTERVALS BETWEEN LARGE RETURNS IN FINANCIAL MARKETS

In the following, we consider historical financial records of the arithmetic returns R_i (1) of the daily closing prices for

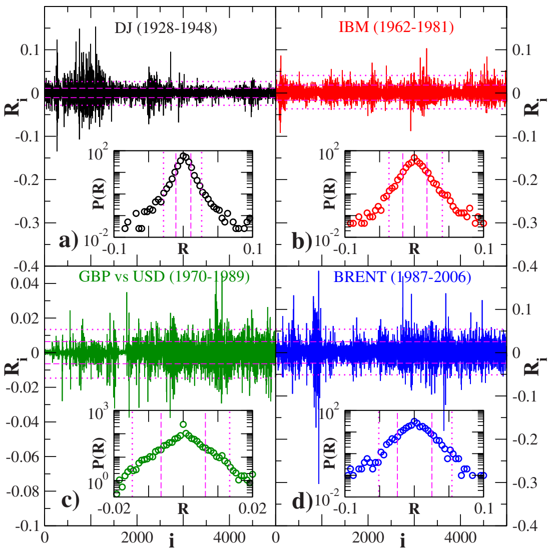

FIG. 6. (Color online) Examples of the arithmetic returns between daily closing prices: (a) Dow Jones index (1928–1948), (b) IBM stock price (1962–1981), (c) British pound versus U.S. dollar exchange rate (1970–1989), and (d) Brent crude oil price (1987–2006). Selected fragments of 5000 data points starting from the beginning of the available data set are shown. Estimates of distribution densities for each of the data series are shown in the relevant insets. Dashed lines show the positive and negative thresholds Q corresponding to the mean return times $R_Q=10$, and dotted lines are for $R_Q=70$; in particular (listed in ascending order) Dow Jones (−0.0284, −0.0106, 0.011, 0.0266), IBM (−0.0365, −0.0166, 0.0183, 0.0405), British Pounds (GBP) vs. U.S. dollar exchange rate (−0.0147, −0.0064, 0.0064, 0.0134), and Brent (−0.0512, −0.0243, 0.0254, 0.0543). The R_Q values have been estimated from the whole record length (from the beginning of the record by 2007), while the examples contain only the first 20 years of each record.

(i) indices, (ii) stocks, (iii) exchange rates, and (iv) commodities obtained from [47–49]. Previous analysis mainly focused on extreme value statistics [50–52], or on the return intervals between volatilities above certain thresholds [4,53–57]. Intertrading times, waiting times between two transactions on the market, as well as exit times have also been analyzed [58–60]. Recently, power-law scaling of waiting times between the consecutive spot price spikes in the Nord Pool electricity market have been reported [61].

It is known that multifractal models are capable of representing several stylized facts in financial records. Here we study whether this multifractality leads to similar effects in the return interval statistics as in the simulated multifractal records. Representative examples of the four types of multifractal financial records we consider are shown in Fig. 6. The figure displays the bursty behavior characteristic for multifractal records and shows also the distribution of the data.

We discuss the results for the return intervals separately for each class of records. We start with the market indices in Fig. 7. Figures 7(a)–7(c) show the PDFs $P_Q(r)$ of the return intervals for $R_Q=10$, 30, and 70, respectively. For comparison, we present also the PDFs for the simulated data, generated for both the MRC and the MRW models for a system size $L=2^{14}$ which is comparable to the length of the available financial records, and averaged over $N=500$ configurations.

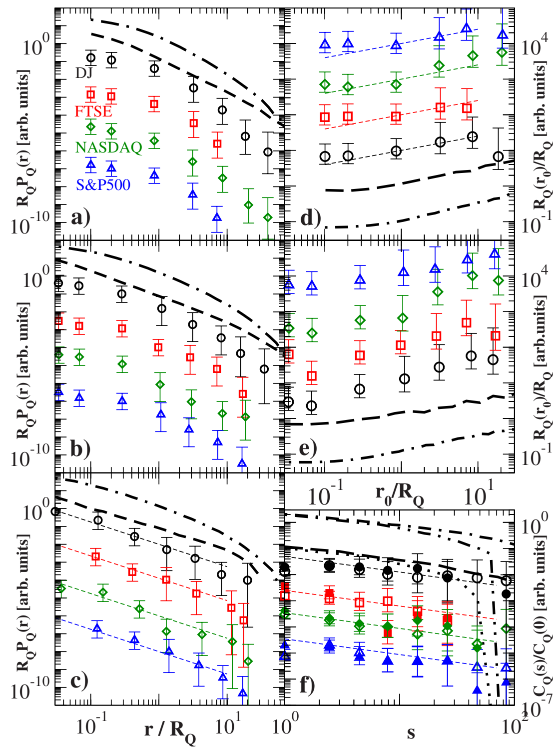

FIG. 7. (Color online) Return interval statistics for the arithmetic returns of the daily closing values of financial indices: Dow Jones (circles), Financial Times Stock Exchange (FTSE) (squares), National Association of Securities Dealers Automated Quotation (NASDAQ) (diamonds), and S&P 500 (triangles), compared with the corresponding quantities for the MRC model (thick dashed lines), and the MRW model (thick dash-dotted lines) of the system size 2^{14}, all given in arbitrary units. (a)–(c) PDFs of the return intervals for $R_Q=10$, 30, and 70, respectively; (d), (e) conditional return periods for $R_Q=10$ and 30, respectively; (f) ACFs of the return intervals for $R_Q=10$ (open symbols for financial records, dashed line for the MRC model, dash-dotted line for the MRW model) and $R_Q=30$ (filled symbols for financial records, dotted line for the MRC model, dash-dot-dotted line for the MRW model). The thin dashed lines in (c), (d), and (f) represent the shifted best fits for the multiplicative cascade model: $P_Q(r) \sim (r/R_Q)^{-1.25}$, $R_Q(r_0) \sim (r_0/R_Q)^{0.4}$, and $C_Q(s) \sim s^{-0.6}$, respectively. The error bars represent the 95% quantiles.

The figure shows that for $R_Q=10$ the PDFs of the return intervals seem to be better approximated by the MRW model, while with increasing R_Q the situation changes. For $R_Q=70$, the PDFs can be nearly perfectly approximated by the best fit $P_Q(r) \sim (r/R_Q)^{-1.25}$ for the MRC model. The figure suggests that, while the MRW model is a better approximation when considering return intervals between events of intermediate magnitude, the MRC model provides a better approximation for the more extreme returns, and may be better suitable for better risk estimations.

Figures 7(d) and 7(e) show the conditional return periods $R_Q(r_0)$ for $R_Q=10$ and 30, again being compared with the relevant quantities for the simulated data. For $R_Q=10$, both the MRC and the MRW models provide nearly a perfect fit for the market data (the best fit for the MRC model, $R_Q(r_0) \sim (r_0/R_Q)^{0.4}$, is provided to compare with the real data re-

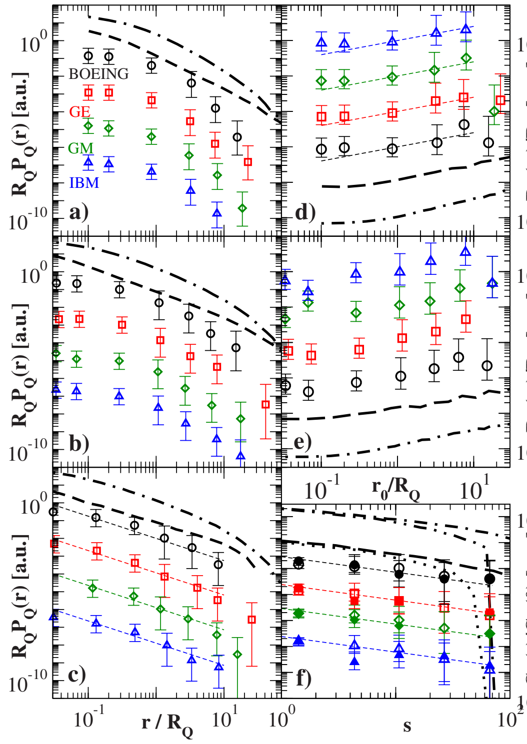

FIG. 8. (Color online) Return interval statistics for the arithmetic returns of daily stock closing prices: Boeing (circles), General Electric (squares), General Motors (diamonds), and International Business Machines (triangles), compared with both multifractal models (the same as in Fig. 7).

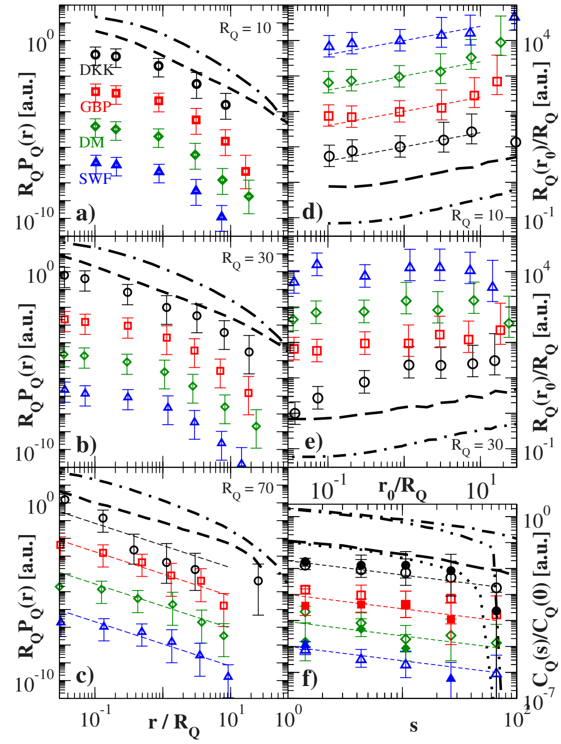

FIG. 9. (Color online) Return interval statistics for the arithmetic returns of the daily closing exchange rates of different currencies versus U.S. dollar: Danish crone (circles), British pound (squares), Deutsche mark (diamonds), and Swiss franks (triangles), compared with both multifractal models (the same as in Fig. 7).

sults). For $R_Q=30$, due to weaker statistics, stronger fluctuations occur.

Finally, Fig. 7(e) shows the ACFs of the return intervals for $R_Q=10$ and 30, which for both models are well approximated by $C_Q(s) \sim s^{-0.6}$.

The corresponding analysis for a number of individual stocks, exchange rates, and commodities, is shown in Figs. 8, 9, and 10, respectively. Quite surprisingly, they all obey qualitatively similar laws as the market indices, which are consistent with the findings for the simulated multifractal data series.

All figures show that the PDFs of the return intervals, for $R_Q=70$, show a perfect power-law decay that is in quantitative agreement with the MRC model. This kind of universal behavior is surprising, since it shows that the very different financial records behave in the same characteristic way. For smaller R_Q values, $R_Q=10$ and 30, there are deviations from this kind of behavior at small scales, which are better described by the MRW model. Again, it is surprising, that all records show the same features.

Also the conditional return periods show the same qualitative behavior for all records. The increase with r_0/R_Q is well described by both models for $R_Q=10$, while for $R_Q=30$ due to the limited statistics larger fluctuations occur, but the results for the financial records agree with the model results. The ACF of the returns, finally, shows the same (universal) behavior for all financial records, which agrees nicely with the prediction of both models. Due to the large fluctuations in the comparatively short financial data series, one cannot observe the small differences in the exponents for both R_Q values, as predicted by the models.

We wish to note that we mainly focused here on the return intervals between large positive returns, quantifying large gains in a unit of time. Figure 11 shows that our results are not limited to the positive returns above some threshold Q, but also hold for negative returns below the threshold $-Q$. The figure shows the same quantities obtained for the return intervals between large negative returns for one representative record from every type considered: Dow Jones index, IBM stock price, British pound vs U.S. dollar exchange rate, and Brent crude oil price. Since the applied multifractal models create records which are symmetric around zero, the results for the simulated data remained unchanged.

VI. CONCLUSION

In summary, we found that the return intervals in the financial records obey nearly quantitatively the same laws as the return intervals in two multifractal models, the MRC and the MRW models. The universal character of the laws (power-law behavior of the PDF of the return intervals and power-law decay of the ACF of the return intervals) is surprising, since the considered financial records differ very strongly, varying from global market indices to exchange rates and oil prices. Since these laws are due to the inherent nonlinear memory in the records, this indicates that the nonlinear memory has a universal characteristic, being the same for all financial records.

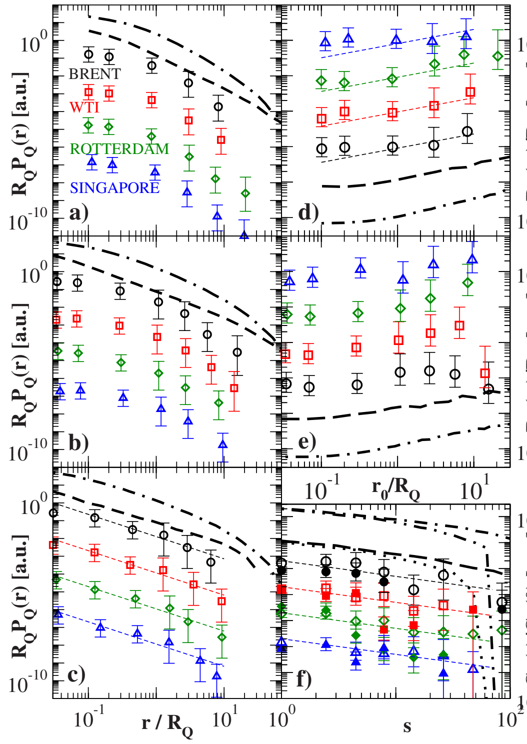

FIG. 10. (Color online) Return interval statistics for the arithmetic returns of the daily closing oil and gasoline prices: Brent crude oil (circles), West Texas Intermediate (WTI) crude oil (squares), Rotterdam gasoline (diamonds), and Singapore gasoline (triangles), compared with both multifractal models (the same as in Fig. 7).

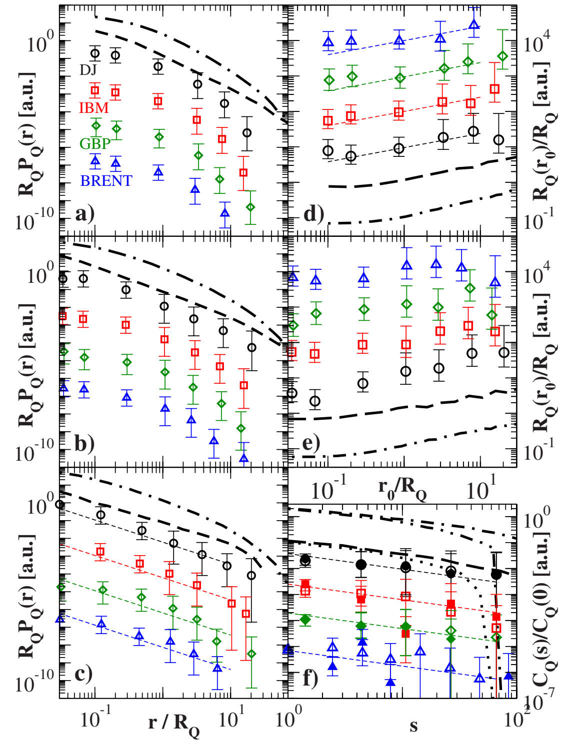

FIG. 11. (Color online) Return interval statistics for large negative arithmetic returns of the daily closing Dow Jones index (circles), IBM stock price (squares), British pound vs U.S. dollar exchange rate (diamonds), and Brent crude oil price (triangles), compared with both multifractal models (the same as in Fig. 7).

In our study, we concentrated on multifractal data series with vanishing linear correlations. Our results are not straightforwardly applicable, for example, to the volatility in financial records, that show multifractal features and in addition are characterized by a linear autocorrelation function that decays by a power law. In this case, linear and nonlinear correlations are superimposed, and this leads, as we have shown recently by extending the MRC model [62,63], to a behavior of the PDF which is characterized by a power-law decay at small scales followed by a (stretched) exponential decay at large scales, which is difficult to distinguish from the behavior expected for long-term correlated records [5]; see also [4,7,57].

ACKNOWLEDGMENTS

We wish to thank our colleagues Jan F. Eichner, Jan W. Kantelhardt, and Shlomo Havlin for valuable discussions.

[1] *The Science of Disasters—Climate Disruptions, Heart Attacks, and Market Crashes*, edited by A. Bunde, J. Kropp, and H.-J. Schellnhuber (Springer, Berlin, 2002).
[2] C. Pfister, *Wetternachhersage: 500 Jahre Klimavariationen und Naturkatastrophen 1496–1995* (Verlag Paul Haupt, Bern, 1998).
[3] R. Glaser, *Klimageschichte Mitteleuropas* (Wissenschaftliche Buchgesellschaft, Darmstadt, 2001).
[4] K. Yamasaki, L. Muchnik, S. Havlin, A. Bunde, and H. E. Stanley, Proc. Natl. Acad. Sci. U.S.A. **102**, 9424 (2005).
[5] A. Bunde, J. F. Eichner, J. W. Kantelhardt, and S. Havlin, Phys. Rev. Lett. **94**, 048701 (2005).
[6] E. G. Altmann and H. Kantz, Phys. Rev. E **71**, 056106 (2005).
[7] J. F. Eichner, J. W. Kantelhardt, A. Bunde, and S. Havlin, Phys. Rev. E **75**, 011128 (2007).
[8] M. I. Bogachev, J. F. Eichner, and A. Bunde, Phys. Rev. Lett. **99**, 240601 (2007).
[9] M. Mudelsee, M. Börngen, G. Tetzlaff, and U. Grünewald, Nature (London) **425**, 166 (2003).
[10] H. E. Stanley, L. A. N. Amaral, A. L. Goldberger, S. Havlin, P. Ch. Ivanov, and C.-K. Peng, Physica A **270**, 309 (1999).
[11] P. Ch. Ivanov, L. A. N. Amaral, Ary L. Goldberger, S. Havlin, M. G. Rosenblum, H. E. Stanley, and Z. R. Struzik, Chaos **11**, 641 (2001).
[12] *Fractals in Biology and Medicine*, edited by G. A. Losa, D. Merlini, T. F. Nonnenmacher, and E. R. Weibel (Birkhäuser, Basel, 2005).
[13] T. Lux and M. Marchesi, Int. J. Theor. Appl. Finance **3**, 475

(2000).

[14] P. Hartmann, S. Straetmans, and C. G. de Vries, in *Asset Price Bubbles: The Implications for Monetary, Regulatory and International Policies*, edited by W. Hunter, G. Kaufman, and M. Pomerleano (MIT Press, Cambridge, MA, 2003).

[15] P. Hartmann, S. Straetmans, and C. G. de Vries, Rev. Econ. Stat. **86**, 313 (2004).

[16] R. H. Riedi, M. S. Crouse, V. J. Ribeiro, and R. G. Baraniuk, IEEE Trans. Inf. Theory **45**, 992 (1999).

[17] D. Helbing, I. Farkas, and T. Viscek, in *The Science of Disasters—Climate Disruptions, Heart Attacks, and Market Crashes* (Ref. [1]), p. 331.

[18] E. Bacry, J. Delour, and J. F. Muzy, Phys. Rev. E **64**, 026103 (2001).

[19] P. A. Samuelson, Industrial Management Review 6, 41 (1965).

[20] E. F. Fama, J. Financ. **25**, 383 (1970).

[21] T. Lux and M. Ausloos, in *The Science of Disasters—Climate Disruptions, Heart Attacks, and Market Crashes* (Ref. [1]).

[22] J. Feder, *Fractals* (Plenum, New York, 1989).

[23] A.-L. Barabasi and T. Vicsek, Phys. Rev. A **44**, 2730 (1991).

[24] *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin (Springer, Berlin, 1991).

[25] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. E. Stanley, Physica A **316**, 87 (2002).

[26] J. F. Muzy, E. Bacry, and A. Arneodo, Phys. Rev. Lett. **67**, 3515 (1991).

[27] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, Nature (London) **381**, 767 (1996).

[28] N. Vandewalle and M. Ausloos, Eur. Phys. J. B **4**, 257 (1998).

[29] K. Ivanova and M. Ausloos, Eur. Phys. J. B **8**, 665 (1999).

[30] N. Vandewalle and M. Ausloos, Int. J. Mod. Phys. C **9**, 711 (1998).

[31] B. B. Mandelbrot, A. Fisher, and L. Calvet, *A Multifractal Model for Asset Returns* (Yale University Press, New Haven, CT, 1997).

[32] A. Fisher, L. Calvet, and B. B. Mandelbrot, *Multifractality of Deutschemark/U.S. Dollar Exchange Rates* (Yale University Press, New Haven, CT, 1997).

[33] A. Arneodo, J.-F. Muzy, and D. Sornette, Eur. Phys. J. B **2**, 277 (1998).

[34] J.-P. Bouchaud, M. Potters, and M. Meyer, Eur. Phys. J. B **13**, 595 (2000).

[35] B. B. Mandelbrot, *Gaussian Self-Affinity and Fractals* (Springer, New York, 2001).

[36] K. Matya, Y. Ashkenazy, and H. E. Stanley, Europhys. Lett. **61**, 422 (2003).

[37] J. Fillol, Econ. Bull. **3**, 1 (2003).

[38] T. Lux, Int. J. Mod. Phys. B **15**, 481 (2004).

[39] T. Lux, Appl. Econ. Lett. **3**, 701 (1996).

[40] F. Wang, K. Yamasaki, S. Havlin, and H. E. Stanley, Phys. Rev. E **77**, 016109 (2008).

[41] S. Taylor, *Modeling Financical Time Series* (John Wiley & Sons, New York, 1986).

[42] Z. Ding, C. W. J. Granger, and R. F. Engle, J. Empirical Finance **1**, 83 (1993).

[43] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Chaos and Fractals: New Frontiers of Science* (Springer, New York, 1992).

[44] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[45] M. Greiner, H. C. Eggers, and P. Lipa, Phys. Rev. Lett. **80**, 5333 (1998).

[46] T. Schreiber and A. Schmitz, Phys. Rev. Lett. **77**, 635 (1996).

[47] http://finance.yahoo.com/

[48] http://research.stlouisfed.org/

[49] http://www.eia.doe.gov/

[50] R. N. Mantegna, Physica A **79**, 232 (1991).

[51] R. N. Mantegna and H. E. Stanley, Nature (London) **376**, 46 (1995).

[52] R. N. Mantegna and H. E. Stanley, Nature (London) **383**, 587 (1996).

[53] T. Kaizoji and M. Kaizoji, Physica A **336**, 563 (2004).

[54] F. Wang, K. Yamasaki, S. Havlin, and H. E. Stanley, Phys. Rev. E **73**, 026117 (2006).

[55] K. Yamasaki, L. Muchnik, S. Havlin, A. Bunde, and H. E. Stanley, *Practical Fruits of Econophysics* (Springer, Tokyo, 2006), p. 43.

[56] J. Masoliver and J. Perelló, Phys. Rev. E **75**, 046110 (2007).

[57] P. Weber, F. Wang, I. Vodenska-Chitkushev, S. Havlin, and H. E. Stanley, Phys. Rev. E **76**, 016109 (2007).

[58] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, Physica A **287**, 468 (2000).

[59] M. Raberto, E. Scalas, and F. Mainardi, Physica A **314**, 749 (2002).

[60] M. Montero, J. Perelló, J. Masoliver, F. Lillo, S. Micciché, and R. N. Mantegna, Phys. Rev. E **72**, 056101 (2005).

[61] J. Perello, M. Montero, L. Palatella, I. Simonsen, and J. Masoliver, J. Stat. Mech.: Theory Exp. (2006) P11011.

[62] M. I. Bogachev, J. F. Eichner, and A. Bunde, Eur. Phys. J. Spec. Top. **161**, 181 (2008).

[63] M. I. Bogachev, J. F. Eichner, and A. Bunde, Pure Appl. Geophys. **165**, 1195 (2008).